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THE BOUNDARY CONDITIONS IN THE TWO-DIMENSIONAL THEORY OF SHELLS. THE MATHEMATICAL ASPECT OF THE PROBLEM†

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The formulation of the boundary conditions based on asymptotic approaches is discussed for the two-dimensional linear static theory of shells. The error of the traditional boundary conditions is estimated in the simplest cases. They are formulated for the first time for more complicated cases. A modified Saint-Venant principle, adapted for use in two-dimensional theories of shells and which eliminates their apparent contradiction, is formulated. Examples of the clamping of the edges, for which the Kelvin–Tait transformation lose their meaning are presented. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We will assume that the three-dimensional elastic medium which forms the shell is referred to a triorthogonal system of coordinates $(\alpha_1, \alpha_2, \alpha_3)$ in which the radius vector of points of the medium are specified by the equation

$$\mathbf{P}(\alpha_1, \alpha_2, \alpha_3) = \mathbf{M}(\alpha_1, \alpha_2) + \alpha_3\mathbf{n}$$

where $\mathbf{M}(\alpha_1, \alpha_2)$ defines the middle surface, $\mathbf{n}$ is the unit vector of the normal, and the faces are specified by the equalities $\alpha_3 = \pm h$. It is assumed that the half-thickness of the shell h is small compared with the characteristic dimension of its middle surface R (for a shell it is convenient to take R to mean the characteristic radius of curvature, while for a plate it is any dimension of its middle plane).

We will assume that the shell has a butt (a narrow boundary), which is not necessarily unique and, to fix our ideas, we will assume that it coincides with the coordinate surface $\alpha_1 = 0$. We will assume that in terms of the three-dimensional theory of elasticity, a boundary-value problem is formulated in which the following conditions must be satisfied: on the faces $\alpha_3 = \pm h$ are conditions which denote that there is no clamping and which specify external forces applied to these surfaces; on the butt $\alpha_1 = 0$ there are three conditions, which determine the form of the clamping (the existence of absence of other butts of the shell plays no part in the later discussion).

We will further discuss mathematical methods of formulating the conditions which must be imposed on the line $\alpha_1 = 0$ of the middle surface in the case considered when analysing the shell using the two-dimensional Kirchhoff–Love type theory. Here it is assumed that no (even obvious) physical considerations must be used.

We will investigate once again a free and rigidly clamped end face from this point of view. Moreover, we will discuss more complex butt conditions, for which the problem of the two-dimensional analogue cannot be solved by elementary methods.

We will use the concept of the separation of the stress–strain state of a thin shell into an internal stress–strain state and a boundary stress–strain state. In statics the latter means the stress–strain state and a boundary stress–strain state. In statics the latter means the stress–strain state localised in the region of the butt of the shell (or other stress concentrators) and which decays exponentially in a certain way with distance from the line $\alpha_1 = 0$. The internal stress–strain state and the boundary stress–strain state of a thin shell differ radically from one another in their properties and will be considered separately here, as is done in the asymptotic theory of the integration of singularly degenerate differential equations.

We will assume that for the internal stress–strain state of the shell, the notion of its variability with respect to the coordinate variables α_1, α_2 has a fairly definite meaning (or, that this internal stress–strain state can conveniently be represented as the sum of terms possessing such a property). Hence, using the method of scale transformations of independent variables, we can put

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$$\alpha_1 = R\lambda^{-p}\xi_1, \quad \alpha_2 = R\lambda^{-p}\xi_2, \quad \alpha_3 = R\lambda^{-l}\zeta \tag{1.1}$$

and we will assume that λ is a large parameter, defined by the formula $\lambda^l = R/h$ (l is an arbitrarily chosen number that is not too large), and that any differentiation with respect to the new independent variables ξ_1, ξ_2, ζ does not change the asymptotic form of the required quantities. The number p in (1.1) is defined by the relation $p/l = t$, in which t is the variability index of the required internal stress–strain state (if the variability of the internal stress–strain state with respect to α_1 and α_2 is different, t must be regarded as the so-called common, i.e. greatest, index).

When determining the boundary stress–strain state we will assume that it is constructed in the region of the coordinate surface $\alpha_1 = 0$ and must possess high variability, both with respect to the variable α_1 (in order that attenuation should occur with distance from $\alpha_1 = 0$), and with respect to the variable α_3 (in order that the face conditions can be satisfied on closely situated surfaces $\alpha_3 = \pm h$). Correspondingly, for the boundary stress–strain state, scale transformations of the independent variables must be carried out using the formulae

$$\alpha_1 = R\lambda^{-l}\theta_1, \quad \alpha_2 = R\lambda^{-p}\theta_2, \quad \alpha_3 = R\lambda^{-l}\zeta \tag{1.2}$$

Here θ_1, θ_2 possess the same properties as ξ_1, ξ_2 in (1.1), while ζ has the previous meaning.

Remark. We will postulate that the boundary stress–strain state has the same large (equal to unity) variability index with respect to α_1 and α_3. We will show below that, when satisfying the butt conditions, this is no way leads to incorrect boundary-value problems (other forms of (1.2) lead to some contradictions).

It was shown in [1–4] that the internal and boundary iterative processes for integrating the differential equations of the three-dimensional theory of elasticity enable one to construct the internal stress–strain state and the boundary stress–strain state of a thin shell separately. In the initial approximation they simultaneously enable one to express the complete stress–strain state of the shell by the following formulae

$$\begin{aligned}
\tau_{ii} &= (1+\alpha_3/R_j)\sigma_{ii} = \lambda^l(\tau_{ii}^0 + \zeta\lambda^{-l+2p-c+b}\tau_{ii}^1) + \tilde{\tau}_{ii} \\
\tau_{ij} &= (1+\alpha_3/R_i)\sigma_{ij} = \lambda^l(\tau_{ij}^0 + \zeta\lambda^{-l+2p-c+b}\tau_{ij}^1) + \tilde{\tau}_{ij} \\
\tau_{i3} &= (1+\alpha_3/R_j)\sigma_{i3} = \lambda^p(\tau_{i3}^0 + \zeta\tau_{13}^1 + \zeta^2\lambda^{-l+2p-c+b}\tau_{i3}^2) + \tilde{\tau}_{i3} \quad (i,j=1,2;\ i\neq j) \\
\tau_{33} &= (1+\alpha_3/R_1)(1+\alpha_3/R_2)\sigma_{33} = \lambda^c(\tau_{33}^0 + \zeta\tau_{33}^1 + \zeta^2\lambda^{-l+2p-c+b}\tau_{33}^2 + \zeta^3\lambda^{-2l+2p-2c+b}\tau_{33}) + \tilde{\tau}_{33}
\end{aligned} \tag{1.3}$$

$$\begin{aligned}
\upsilon_i &= \lambda^{l-p+b}(\upsilon_i^0 + \zeta\lambda^{-l+2p-c}\upsilon_i^1) + \tilde{\upsilon}_i \quad (i=1,2) \\
\upsilon_3 &= \lambda^{l-c+b}(\upsilon_3^0 + \zeta\lambda^{-l+c}\upsilon_3^1) + \tilde{\upsilon}_3 \\
(\tilde{\tau}_{mn} &= \lambda^{\mu+\alpha}ES_{mn}(\alpha) + \lambda^{\nu+\beta}ES_{mn}(\beta), \\
\tilde{\upsilon}_m &= \lambda^{\mu+\alpha-l}RV_m(\alpha) + \lambda^{\nu+\beta-l}RV_m(\beta), \quad m,n=1,2,3)
\end{aligned} \tag{1.4}$$

Here σ_{mn}, v_m (m, n = 1, 2, 3) mean the stresses and displacements of a three-dimensional elastic medium, while τ_{mn} are the so-called asymmetrical stresses, the meaning of which follows from (1.3), and R_k are the principal radii of curvature of the middle surface.

The asymptotic factors λ^l and λ^p in (1.3) and (1.4) have the same meaning as in (1.1) and (1.2). We will say more about λ^c, λ^b below. The quantities denoted by τ, υ (when this causes no confusion, here and below we will only write the root letters, i.e. we will drop all the superscripts and subscripts), are functions of the two variables ξ_1, ξ_2 or, which is the same thing, α_1, α_2. They define the stresses (τ) and the displacements (υ) of the internal stress–strain state and are the required functions of the theory of thin shells. They are related to the required quantities of this theory by the formulae

$$\begin{aligned}
&\tau_{ii}^0 = \frac{1}{2R}T_i, \quad \tau_{ij}^0 = \frac{1}{2R}S_{ij}; \quad \tau_{ii}^1 = -\lambda^{2l-2p+c-b}\frac{3}{2R^2}G_i \\
&\tau_{12}^1 = \lambda^{2l-2p+c-b}\frac{3}{2R^2}H_{12}, \quad \tau_{i3}^0 + \frac{1}{3}\lambda^{-l+2p-c+b}\tau_{i3}^2 = -\frac{\lambda^{l-p}}{2R}N_i \\
&\upsilon_i^0 = \lambda^{-l+p-b}u_i, \quad \upsilon_3^0 = -\lambda^{-l+c-b}w, \quad \upsilon_i^1 = -R\lambda^{-l-p+c-b}\gamma_i
\end{aligned} \tag{1.5}$$

In these equations we have used the notation of monograph [4] for quantities on the right. These quantities satisfy all the equations of the general two-dimensional theory of shells derived there (this includes the adequacy of the single-term asymptotic form of the internal stress–strain state and of the Kirchhoff–Love theory).

The following additional relations [4] (not stipulated in the classical theory of shells) also hold

$$\tau_{i3}^0 + \lambda^{-l+2p-c+b}\tau_{i3}^2 = \lambda^{-p}(\tau_{i3}^+ - \tau_{i3}^-)/2 \tag{1.6}$$

$$\tau_{i3}^1 = \lambda^{-p}(\tau_{i3}^+ - \tau_{i3}^-)/2 \quad (i = 1,2), \quad v_3^1 = -\lambda^{-b}\nu(T_1 + T_2)/(2E)$$

In their derivation, described in [4], it is assumed that the conditions on the faces have the form

$$\alpha_3 = \pm h\colon\ \tau_{k3} = \tau_{k3}^\pm \quad (k = 1,2,3)$$

We have denoted by S and V in (1.3) and (1.4) the dimensionless stresses and displacements of the boundary stress–strain state (ES are dimensional asymmetrical stresses, hV are dimensional displacements, and the subscripts and superscripts are also omitted here and below). We have taken into account the fact that the boundary stress–strain state is separated into an antiplane stress–strain state, approximately defined in the (θ_1, ζ) plane by the equations of the antiplane problem of the theory of elasticity, while the plane stress–strain state is defined in the same way by the equations of the plane problem. The quantities S and V are marked with the additional letters (α) and (β) respectively. This means [1, 4] that among $S_{ij}(\alpha)$ and $V_k(\alpha)$ the following quantities are asymptotically principal quantities

$$P = [S_{12}(\alpha), S_{23}(\alpha), V_2(\alpha)] \tag{1.7}$$

while among $S_{ij}(\beta)$ and $V_k(\beta)$ the following quantities are the principal quantities

$$Q = [S_{11}(\beta),\ S_{22}(\beta),\ S_{33}(\beta),\ S_{13}(\beta),\ V_1(\beta),\ V_3(\beta)] \tag{1.8}$$

This property of the boundary stress–strain state is taken into account in (1.3) and (1.4) by the asymptotic factors $\lambda^\mu\, \lambda^\nu$ for the quantities S and V. When we are concerned with the quantities P we must put $\mu = 0$, $\nu = -l + p$, and for the quantities Q we must assume $\mu = -l + p$, $\nu = 0$ [4].

The antiplane and plane problems when constructing the quantities $S(\alpha)$, $V(\alpha)$ and $S(\beta)$, $V(\beta)$ must be solved taking into account the homogeneous face conditions, since the external forces applied to the surfaces $\alpha_3 = \pm h$ have already been borne in mind in the internal iterative process. Hence, for the quantities $S(\alpha)$, $V(\alpha)$ and $S(\beta)$, $V(\beta)$ in (1.3) and (1.4) we have introduced the asymptotic factors λ^α λ^β in which α, β must be determined depending on the form of the butt conditions using the discussions described below.

Formulae (1.3) and (1.4) define, in explicit form, the asymptotic form of a certain family of integrals of the differential equations of the three-dimensional theory of elasticity of a thin body with free faces. It follows from the results of the following sections of this paper that the corresponding stress–strain state may be approximately subject to not only the face conditions but also the butt conditions of the three-dimensional theory of elasticity (an iterative improvement of this result is also possible). Hence it follows that the powers λ in (1.3) and (1.4) specify the asymptotic form of the complete stress–strain state of the shell. It depends on the parameters η, t, c and b, the physical meaning of which is as follows: η is the dimensionless half-thickness (the principal parameter of the shell asymptotic form), $p/l = t$ is the variability index of the internal stress–strain state of the shell and c is the discriminant parameter; it is related to the parameters p and l by the following relations.

For a shell which does not degenerate into a plate

$$c = 0 \ \text{ for } 0 \leqslant t \leqslant \tfrac{1}{2}; \quad c = 2p - l \ \text{ for } t \geqslant \tfrac{1}{2} \tag{1.9}$$

for a plate

$$c = 2p - l \ \text{ for any } t \tag{1.10}$$

(the parameter c appears in (1.3) and (1.4) due to the fact that when the variability index t passes through the value 1/2, the internal stress–strain state undergoes a qualitative change), and b is the so-called pseudo-bending index [5]; it represents the asymptotic closeness of the deformation of the middle surface of the shell to infinitesimal bendings.

2. A SHELL WITH A FREE BUTT

Suppose the shell has a free butt $\alpha_1 = 0$ on which, by virtue of (1.3), the following approximate conditions of the three-dimensional theory of elasticity must be satisfied

$$\begin{aligned}
&\lambda^l(\tau_{11}^0 + \zeta\lambda^{-l+2p-c}\tau_{11}^1) + \lambda^{-l+p+\alpha}ES_{11}(\alpha) + \lambda^\beta ES_{11}(\beta) = 0 \\
&\lambda^l(\tau_{12}^0 + \zeta\lambda^{-l+2p-c}\tau_{12}^1) + \lambda^\alpha ES_{12}(\alpha) + \lambda^{-l+p+\beta}ES_{12}(\beta) = 0 \\
&\tau_{13}^0 + \tfrac{1}{3}\lambda^{-l+2p-c}\tau_{13}^2 + \zeta\tau_{13}^1 + \lambda^{-l+2p-c}(\zeta^2 - \tfrac{1}{3})\tau_{13}^2 + \lambda^{-l+\alpha}ES_{13}(\alpha) + \lambda^{-p+\beta}ES_{13}(\beta) = 0
\end{aligned} \tag{2.1}$$

In these equations the exponents μ and ν are chosen as described in Section 1, and to reduce the number of versions we have assumed (as everywhere henceforth) that $b = 0$. The superscripts p, l and c in (2.1) are assumed to be given, taking into account the proposed properties of the solution of the problem considered.

The proposed approach to the formulation of the boundary conditions in the classical two-dimensional theory of shells consists of the fact that relations of the form (2.1) are treated as the butt conditions for the problem of constructing the boundary stress–strain state, i.e. the quantities S and V are assumed to be the required quantities, while the quantities τ are regarded for the present as known, and we consider the problem of those conditions which τ must obey on the line $\alpha_1 = 0$ in order that S and V should possess the property of Saint-Venant decay with respect to α_1.

The boundary stress–strain state must be constructed in the half-strip $\{0 \leqslant \alpha_1 < \infty; -h \leqslant \alpha_3 \leqslant +h\}$, taking into account the homogeneous face conditions $S_{3k} = 0$ ($k = 1, 2, 3$) on the line $\alpha_3 = \pm h$ and with the additional requirements on the decay of the three stresses S_{1k} and three displacements V_k as $\alpha_1 \to \infty$. In addition, of course, we must satisfy the condition that the boundary stress–strain state must be bounded as $\lambda \to \infty$. To satisfy these requirements in (2.1) we can arrange the values of the weighting exponents α and β and prescribe the form of the four conditions set out in the classical theory of shells on the line $\alpha_1 = 0$ for the values of τ Hence, we must obtain that:

(a) in relations (2.1), after dropping the common factors of the form $\lambda^\varkappa$, it should be possible to take the limit as $\lambda \to \infty$ (i.e. there are no positive powers of λ);

(b) from the limiting butt relations (as $\lambda \to \infty$) one butt condition follows for the antiplane problem and two butt conditions for the plane problem;

(c) the limiting butt conditions for both the antiplane problem and the plane problem separately do not admit of a trivial (zero) solution (otherwise this would indicate that there is an error in the choice of α, β).

These requirements will be satisfied if the following relations are satisfied

$$\alpha_1 = 0:\ \tau_{11}^0 = 0,\ \ \tau_{12}^0 = 0,\ \ \tau_{11}^1 = 0 \tag{2.2}$$

and if the following formulae are assumed for the weighting factors α, β

$$\alpha = 2p - c,\ \ \beta = p \tag{2.3}$$

Here the limit boundary relations (2.1) take the form

$$\begin{aligned}
&\lambda^{-l+2p-c}ES_{11}(\alpha) + ES_{11}(\beta) = 0,\ \ \zeta\tau_{12}^1 + ES_{12}(\alpha) = 0 \\
&\tau_{13}^0 + \tfrac{1}{3}\lambda^{-l+2p-c}\tau_{13}^2 + \zeta\tau_{13}^1 + \lambda^{-l+2p-c}(\zeta^2 - \tfrac{1}{3})\tau_{13}^2 + \\
&+\lambda^{-l+2p-c}ES_{13}(\alpha) + ES_{13}(\beta) = 0
\end{aligned} \tag{2.4}$$

In (2.4) we have retained terms with the factor $\lambda^{-l+2p-c}$, although we have assumed that the limit was taken as $\lambda \to \infty$. The point is that this factor, according to (1.9) and (1.10), is either negative (when $p < l/2$), or is equal to zero (when $p \geqslant l/2$). It must be interpreted in the appropriate way depending on the values of the parameters p and l.

It can be assumed that the inhomogeneity requirements are also satisfied in (2.4). This, generally speaking, is ensured by the terms with τ_{12}^1, τ_{13}^0, τ_{13}^2, τ_{13}^1 in (2.4). According to (1.5) and (1.6) they do not belong to quantities which, in the two-dimensional theory of shells, are subject to any boundary requirements. Hence, the free terms in (2.4) can only vanish "accidentally", i.e. only in specific problems for specific value of the input data.

The problem remains of ensuring the Saint-Venant decay of the boundary stress–strain state. We will examine this problem after considering some general properties of the approximate theory of the boundary stress–strain state in the following section.

3. A MODIFIED SAINT-VENANT PRINCIPLE IN THE THEORY OF SHELLS

We will consider the boundary stress–strain state of a shell in the vicinity of the butt $\alpha_1 = 0$ and we will write the equilibrium equations of the three-dimensional theory of elasticity for it in the following form

$$X_i \equiv \frac{\partial\sigma_{i1}}{\partial\alpha_1} + \frac{\partial\sigma_{i2}}{\partial\alpha_2} + \frac{\partial\sigma_{i3}}{\partial\alpha_3} = 0 \quad (i = 1,2,3) \tag{3.1}$$

(for simplicity we have used Cartesian coordinates, but the final conclusions remain true for any coordinate system).

We replace the required quantities in (3.1) using the formulae

$$\sigma_{st} = ES_{st} \quad (s, t = 1, 2, 3) \tag{3.2}$$

and for the independent variables we make the replacement (1.2) and take into account the fact that the asymptotic form of the boundary stress–strain state can be expressed by the relations [4]

$$\begin{aligned}\frac{1}{E}(\sigma_{11}, \sigma_{22}, \alpha_{33}, \sigma_{13}) &= \lambda^{\beta}[S_{11}(\beta), S_{22}(\beta), S_{33}(\beta), S_{13}(\beta)] + \\ &+ \lambda^{\alpha-l+p}[S_{11}(\alpha), S_{22}(\alpha), S_{33}(\alpha), S_{13}(\alpha)]\end{aligned} \tag{3.3}$$

$$\frac{1}{E}(\sigma_{12}, \sigma_{23}) = \lambda^{\alpha}[S_{12}(\alpha), S_{23}(\alpha)] + \lambda^{\beta-l+p}[S_{12}(\beta), S_{23}(\beta)] \tag{3.4}$$

In these equations $S(\alpha)$, $S(\beta)$ are quantities of the form $O(\lambda^{\varkappa})$ for the same $\varkappa$ for all S, and α, β are weighting factors, introduced in Section 1. The latter characterize the relative asymptotic intensity of the antiplane (α) and plane (β) stresses in the boundary stress–strain state.

Relations (3.3) and (3.4) were derived neglecting quantities of the form

$$\varepsilon = O(\lambda^{-l+p}) \tag{3.5}$$

In the discussions relating to the formulation of the boundary conditions, we assumed that the neglect of other quantities of the same order was acceptable.

We will agree to consider the following three cases separately

$$0 < \alpha - \beta < l - p \tag{3.6}$$

$$0 < \beta - \alpha < l - p \tag{3.7}$$

$$\alpha - \beta = 0 \tag{3.8}$$

(the quantities α, β, which do not fit into these frameworks, will not be necessary below).

Boundary stress–strain states with different asymptotic properties are characterized by relations (3.6)–(3.8). We will assume that the following are possible: an almost antiplane boundary stress–strain state, in which inequalities (3.6) are satisfied and σ_{12} and σ_{23} are asymptotically continuous, an almost plane boundary stress–strain state, in which inequalities (3.7) are satisfied and σ_{11}, σ_{22}, σ_{33}, σ_{13} are continuous, and a mixed boundary stress–strain state, in which equality (3.8) holds and all these stresses are asymptotically equivalent.

It can be verified that when any of relations (3.6)–(3.8) are satisfied, it is sufficient to retain only the first terms on the right-hand sides of (3.3) and (3.4) to within the accuracies specified by (3.5). Moreover, scale transformations (1.2) hold for the boundary stress–strain state. This enables us to conclude that, with the accuracy of (3.5), the equilibrium equations (3.1) can be replaced by approximate equalities, which have the form

$$X_1 \equiv \frac{\partial\sigma_{11}}{\partial\alpha_1} + \left(\frac{\partial\sigma_{12}}{\partial\alpha_2}\right)_{(3.6)} + \frac{\partial\sigma_{13}}{\partial\alpha_3} = 0$$

$$X_2 \equiv \frac{\partial\sigma_{21}}{\partial\alpha_1} + \left(\frac{\partial\sigma_{22}}{\partial\alpha_2}\right)_{(3.7)} + \frac{\partial\sigma_{23}}{\partial\alpha_3} = 0 \tag{3.9}$$

$$X_3 \equiv \frac{\partial\sigma_{31}}{\partial\alpha_1} + \left(\frac{\partial\sigma_{32}}{\partial\alpha_2}\right)_{(3.6)} + \frac{\partial\sigma_{33}}{\partial\alpha_3} = 0$$

We have assumed here that the terms in brackets need to be retained only when the relation, the number of which is indicated outside the brackets, is satisfied. (Thus, when (3.8) is satisfied all the terms in the brackets drop out.)

We will introduce into consideration the following four obvious equations

$$\iint X_1 dg = 0, \quad \iint X_2 dg = 0, \quad \iint X_3 dg = 0$$
$$\iint (X_1\alpha_3 - X_3\alpha_1) dg = 0 \tag{3.10}$$

in which the integration is extended over the region $g = \{0 \leqslant \alpha_1 \leqslant \infty; -h \leqslant \alpha_3 \leqslant +h\}$ and will expand their left-hand sides by traditional methods of the theory of elasticity, taking into account the fact that the stresses σ_{st} must be subject to the homogeneous face conditions

$$\alpha_3 = \pm h\colon \sigma_{3k} = 0 \quad (k = 1, 2, 3)$$

and the butt conditions

$$\alpha_1 = 0\colon \ \sigma_{11} = \lambda^l s_{11}, \quad \sigma_{12} = \lambda^l s_{12}, \quad \sigma_{13} = \lambda^p s_{13} \tag{3.11}$$

(s_{11}, s_{12} and s_{13} are specified functions of the variables α_2 and α_3, commensurable with λ^0) and the conditions for sufficiently rapid decay of the stresses σ as $\alpha_1 \to \infty$.

The corresponding actions were described in detail in [6] for the case when the approximate equilibrium equations correspond to conditions (3.6). Here Eqs (3.9), taking (3.11) into account, can be reduced to four equations (integration over α_3 here and later is carried out from $-h$ to h)

$$\int s_{11} d\alpha_3 = 0, \quad \int s_{12} d\alpha_3 = 0$$
$$\lambda^p \int s_{13} d\alpha_3 + \lambda^l \frac{\partial}{\partial\alpha_2} \int s_{12}\alpha_3 d\alpha_3 = 0 \tag{3.12}$$
$$\lambda^l \int s_{11}\alpha_3 d\alpha_3 + \left\{\frac{\partial}{\partial\alpha_2} \iint \sigma_{12}\alpha_3 dg\right\} = 0$$

In the last of these, the term in the braces is negligibly small, to within the accuracy described by (3.5). This follows from (3.11) and from the fact that the additional integration over α_1 of the exponentially decaying function σ_{12} is provided for in it.

After dropping the small term, (3.12) will contain only values of the stresses specified on the line $\alpha_1 = 0$, and, consequently, conditions (3.12) can be formulated without solving the corresponding boundary-value problem. These conditions are obviously necessary for a decaying solution of the boundary-value problem in question to exist. Here we will mean by the modified Saint-Venant principle, as previously [6], the assertion that conditions of this kind are also sufficient.

After obvious reduction using (1.5), Eqs (3.12), taken without the term in the braces, have the form

$$T_1 = S_{21} = N_1 + \frac{\partial H_{21}}{\partial\alpha_2} = G_1 = 0 \text{ when } \alpha_1 = 0 \tag{3.13}$$

They confirm the boundary conditions in the classical two-dimensional theory of shells for a free boundary, generally accepted at the present time. The error in these conditions is governed by estimate (3.5) used in this section. Below we will show that this is characteristic, generally speaking, for traditional boundary conditions of the two-dimensional theory of shells. Nevertheless, it has been shown [4] that, when deriving the two-dimensional differential equations of the theory of shells, a higher accuracy of the order of λ^{-2l+2p} is easily achieved. The question naturally arises of refining the formulation and boundary conditions to quantities of the same order (these conditions have been called reduced conditions). However, the above discussion shows that for the boundary conditions the question of refinement is more complex than for differential equations, and the construction of more accurate terms involves the need to solve some additional boundary-value problems of the plane theory of elasticity.

Remark. It can be shown that the accuracy (3.5) achieved here is sufficient to justify the use of relations (2.2) in deriving limit inequality (2.4).

We will now assume that, instead of (3.6), inequalities (3.7) are satisfied. Using the approximate equilibrium equations (3.9), we obtain from the first, third and fourth relations of (3.10)

$$\int s_{11} d\alpha_3 = \int s_{13} d\alpha_3 = \int s_{11}\alpha_3 d\alpha_3 = 0$$

To convert the second equation of (3.10) we will take into account the fact that in the case considered the plane boundary layer is the main one for which the following approximate equation holds [4]

$$\sigma_{22} + \nu(\sigma_{11} + \sigma_{33}) = 0$$

and the expression for X_2 in (3.9) reduces to the form

$$X_2 \equiv \frac{\partial\sigma_{21}}{\partial\alpha_1} - \nu\left(\frac{\partial\sigma_{11}}{\partial\alpha_2} + \frac{\partial\sigma_{33}}{\partial\alpha_2}\right) + \frac{\partial\sigma_{23}}{\partial\alpha_3} \tag{3.14}$$

When interpreting the second equation of (3.10) the only difficulties that arise are those related to the second term in (3.14). We will carry out appropriate calculations (integration with respect to α_1 is carried out from zero to infinity)

$$\iint \sigma_{11} dg = \int d\alpha_3 \int \sigma_{11} d\alpha_1 = \int\left\{[\alpha_1\sigma_{11}]_{\alpha_1=0}^{\alpha_1=\infty} - \int \alpha_1 \frac{\partial\sigma_{11}}{\partial\alpha_1} d\alpha_1\right\} d\alpha_3$$

The first term in the braces vanishes due to the assumption of the exponential decay of the stress σ_{11}. The second term can be converted using the first equation of (3.9). We obtain

$$\iint \sigma_{11} dg = -\iint \alpha_1 \frac{\partial\sigma_{11}}{\partial\alpha_1} dg = \int \alpha_1 d\alpha_1 \int \frac{\partial\sigma_{13}}{\partial\alpha_3} d\alpha_3 = 0$$

Further we have

$$\iint \sigma_{33} dg = \int\left\{[\alpha_3\sigma_{33}]_{\alpha_3=-h}^{\alpha_3=+h} - \int \alpha_3 \frac{\partial\sigma_{33}}{\partial\alpha_3} d\alpha_3\right\} d\alpha_1 = -\iint \alpha_3 \frac{\partial\sigma_{33}}{\partial\alpha_3} dg =$$

$$= \iint \alpha_3 \frac{\partial\sigma_{31}}{\partial\alpha_1} dg = -\lambda^p \int s_{31}\alpha_3 d\alpha_3$$

Finally the conditions for the modified Saint-Venant principle to be satisfied in the case of (3.7), i.e. for a fairly rigidly clamped butt [7], take the form

$$\int s_{11} d\alpha_3 = \int s_{13} d\alpha_3 = \int s_{11}\alpha_3 d\alpha_3 = 0$$

$$\lambda^l \int s_{12} d\alpha_3 + j\nu\lambda^p \frac{\partial}{\partial\alpha_2} \int s_{13}\alpha_3 d\alpha_3 = 0 \tag{3.15}$$

(j is a conditional factor which up till now has been assumed to be equal to unity).

Remarks 1. There is an obvious analogy between the last equation of (3.15) and the third equation of (3.12). The latter, in the classical theory of shells, arises due to the use of the reduced shearing force. However, it does not follow from the last equation of (3.15) that for sufficiently rigid clamping of the butt similar corrections must be introduced for the shearing force. When using (1.2) it can be shown that the term discussed in (3.15) can be dropped with an error of the order of λ^{-2l+2p}. This means that we can put $j = 0$ in (3.15), i.e. we can assume that for sufficiently rigid clamping of the butt, to obtain the modified conditions for Saint-Venant decay we only need to drop the requirement that the boundary twisting moment should vanish in the canonical conditions without changing the meaning of the shearing force.

2. It follows from the above, in particular, that the well known purely static discussions of Kelvin–Tait are not a comprehensive justification for using the reduced shearing force. One needs to take into account not only the statics but also the nature of the clamping of the butt.

It can be verified that the decay conditions (3.15) when $j = 0$ also hold when $\alpha = \beta$.

Note that the proposed approach to formulating the boundary conditions enables one to obtain, in passing, relations of the form (2.3) for the weighting exponents α, β, i.e. to determine the qualitative pattern of boundary elastic phenomena in almost two-dimensional bodies.

Formulae (1.3) show that when $\alpha = \beta = l$, the antiplane boundary layer, the plane boundary layer and the internal stress–strain state of the shell are asymptotically equivalent to one another as regards the intensity of the stresses. In view of this, taking (2.3), (1.9) and (1.10) into account, we conclude that, in the region of the free butt

(a) in a shell (which does not degenerate into a plate) when $t < 1/2$, the boundary stress–strain state as a whole is asymptotically secondary, and when $t \geqslant 1/2$ it is commensurable with the internal stress–strain state;

(b) in a plate the internal stress–strain state and the boundary stress–strain state are of the same order for any t;

(c) in the separately taken boundary stress–strain state, generally speaking, the antiplane boundary layer predominates, and the only exception is the case when the shell does not degenerate into a plane and $t = 0$, i.e. when the boundary stress–strain state is secondary as a whole.

4. CLAMPED BUTTS

We will now consider the case when the butt $\alpha_1 = 0$ is clamped and the conditions of the three-dimensional theory of elasticity

$$\alpha_1 = 0\colon\ v_1 = 0,\ v_2 = 0,\ v_3 = 0 \tag{4.1}$$

are imposed on it.

We will use relations (1.4), putting $b = 0$ in them, we choose μ, ν, as was done in Section 1, and we specify the weighting factors by the formulae

$$\alpha = p,\ \ \beta = l \tag{4.2}$$

We obtain the butt conditions

$$\begin{aligned} &RV_1(\beta) + R\lambda^{-2l+2p}V_1(\alpha) = -\lambda^{l-p}(\mathrm{v}_1^0 + \lambda^{-l+2p-c}\zeta\mathrm{v}_1^1) \\ &RV_2(\alpha) + RV_2(\beta) = -\lambda^{2l-2p}(\mathrm{v}_2^0 + \lambda^{-l+2p-c}\zeta\mathrm{v}_2^1) \\ &RV_3(\beta) + R\lambda^{-2l+2p}V_3(\alpha) = -\lambda^{l-c}(\mathrm{v}_3^0 + \lambda^{-l+c}\zeta\mathrm{v}_3^1) \end{aligned} \tag{4.3}$$

which will once again be regarded as the boundary conditions for the boundary stress–strain state.

Remark. Equations (4.3) were derived previously in [7, 8] without describing the corresponding calculations and contain an error, albeit unimportant for the final results: on the left-hand sides of the first and last equations (4.3), instead of the factors λ^{-2l+2p} the factors $\lambda^{-l+2p-c}$ are erroneously written. Moreover, the expression in brackets on the right-hand side of the last equation (4.3) was supplemented with the term $\lambda^{-2l+2p}\zeta^2\mathrm{v}_3^2$, which was necessary for a correct calculation of the boundary stress–strain state. In the present paper, where we are only concerned with the problem of the boundary conditions in the two-dimensional theory of shells, this refinement is unimportant.

On the left-hand sides of (4.3) all the powers of λ are non-positive. On the right-hand sides λ are positive for the five quantities v_1^0, v_1^1, v_3^0, v_2^0, v_2^1. The first four of these, according to (1.5), are proportional to the displacements u_1, u_2 and w of the middle surface and the elastic rotation angle γ. Hence, it is natural to assume that when $\alpha_1 = 0$ they vanish with asymptotic accuracy, sufficient to neutralize the positive powers of λ. This is confirmed by the results obtained below. A unique positive power of λ, in front of v_2^1, remains, i.e. with a value proportional, by (1.5), to the rotation angle γ_2. The following formula holds for the latter [4]

$$\gamma_2 = -\frac{1}{A_2}\frac{\partial w}{d\alpha_2} - \frac{v_2}{R_2} \tag{4.4}$$

It follows from this that along the line $\alpha_1 = 0$ the quantity v_2^1 vanishes together with v_1^0, v_1^1, v_3^0, v_2^0.

Hence, there is no contradiction in the values of (4.2) for α, β as regards the fact that there are no positive powers of λ in the butt conditions (4.3).

Relations (4.2), generally speaking, also do not contradict the requirement of inhomogeneity, since the third of equations (4.3) contains the terms v_3^1, which, by (1.6), can vanish only "accidentally" when $\alpha_1 = 0$. In this situation (4.2) is subject to review, on which we will not dwell here.

Remark. For a clamped butt essentially the same discrepancy occurs as for a free end; we would expect five rather than four boundary conditions to be imposed on υ. However, in this case, the boundary equation $v_2^1 = 0$, follows from the assumed vanishing of the boundary values of the quantities v_3^0, v_2^0.

Formulae (4.2) show that, in the vicinity of a clamped end the boundary stress–strain state of a shell is always determined mainly by the solution of the plane problem and is of the same order as the internal stress–strain state. Correspondingly, requirements (3.15) with $j = 0$ become the conditions for the applicability of the modified Saint-Venant principle in this case. They must relate to the reaction forces which arise on the butt $\alpha_1 = 0$, i.e. to quantities which are unknown at the stage when the shell boundary conditions are formulated. To overcome this obstacle we introduce into consideration typical problems of the theory of the boundary stress–strain state. We will assume in these that the approximate equations of the anti-plane and plane boundary layers are solved in a rectangle

$$\{0 \leqslant \theta_1 \leqslant \theta_1^0, -1 \leqslant \zeta \leqslant +1\} \tag{4.5}$$

(θ_1^0 is a fairly large number) and we take into account the face conditions

$$\zeta = \pm 1: \quad S_{3j} = 0 \quad (j = 1,2,3)$$

the clamping conditions on a sufficiently distant butt

$$\theta_1 = \theta_1^0: \quad V_i = 0 \quad (i = 1,2,3)$$

and the conditions on the butt considered

$$\theta_1 = 0: \quad V_n = \delta_{mn}\zeta^k \quad (k = 0,1; m,n = 1,2,3)$$

(δ_{mn} is the Kronecker delta).

Hence, problems of the approximate theory of the boundary stress–strain state, solved in a rectangle (4.5) taking the three static or kinematic conditions on each of the rectilinear parts of the boundary into account, are typical. Here, in any specific typical problem, only one condition, defining, when $\theta_1 = 0$, some of the dimensionless displacements as a quantity which varies over the thickness as ζ^0 or ζ^1, is homogeneous.

It is assumed that typical problems can be solved approximately using the procedure described below. If a unique inhomogeneous condition of a typical problem specified the end displacement V_1 or V_3, the quantities Q, defined by relations of the form (3.3) are constructed as a solution of the plane problem of the theory of elasticity (taking into account the related boundary conditions, including also the unique inhomogeneous condition). The corresponding quantities P in this typical problem will be asymptotically secondary (containing, by (3.4), the additional factor λ^{-l+p}). The equations of the inhomogeneous antiplane problem, in which the right-hand sides contain the proportionality factor λ^{-l+p}, serve to construct them. These equations must be solved taking into account the remaining unused homogeneous boundary conditions. Hence, in this case, we can put $P = \lambda^{-l+p} P^*$ and assume that, in formulating the boundary-value problems defining P^* and Q, the large parameter λ does not occur in it, i.e. P^* and Q are commensurable with λ^0.

Equally, for the case when the end displacement V_2 is specified by a unique inhomogeneous boundary condition in a typical problem, we can assume that its solution is determined by the quantities P and $Q = \lambda^{-l+p} Q^*$, in which P and Q^* are of the same order as λ^0.

We will introduced the notation $s_{ij}(V_r = \zeta^k)$ and we will assume that $s_{ij}[\,\cdot\,]$ is the butt value of the dimensional stresses σ_{ij} (when $\theta_1 = 0$), defined by the solution of the typical problem in which the unique inhomogeneous condition is expressed by the equation written in square brackets. For example, the symbol $s_{11}[V_1 = \zeta^1]$ denotes the stress σ_{11} on the butt $\theta_1 = 0$, which arises in the elastic rectangle (4.5) as a result of the application of a displacement V_1 equal to ζ^1 to its butt.

The boundary stress–strain state corresponding to condition (4.3) can be constructed as follows: On the right-hand sides of (4.3) we only retain one term, (for example, $-\lambda^{l-c}\mathrm{v}_2^1$ on the right-hand side of the second butt condition) and noting that in this term the quantity $\lambda^{l-c}\mathrm{v}_2^1|_{\alpha_1=0}$ can depend only on the variable ξ_2, we replace this factor by unity. We obtain one of the typical problems, the solution of which is denoted by the notation $[V_1 = \zeta]$. It occurs in the final expression for the required reactive force with the factor $-\lambda^{l-c}\mathrm{v}_2^1$. Using the same method we also take into account all the remaining terms on the right-hand sides of the end equations (4.3). Here, of course, we must bear in mind that if the inhomogeneity only occurs, for example, in the antiplane problem, all the solutions of the plane problem must be multiplied by the small quantity λ^{-l+p}.

Hence, the reactive forces $s_{lr}[\cdot]$ ($r = 1, 2, 3$) on the butt $\alpha_1 = 0$ are expressed by the formulae

$$Rs_{1j} = -\lambda^{l-p}s_{1j}[V_1 = 1]\mathrm{v}_1^0 - \lambda^{p-c}s_{1j}[V_1 = \zeta]\mathrm{v}_1^1 - \lambda^{l-c}s_{1j}[V_3 = 1]\mathrm{v}_3^0 -$$

$$-\lambda^0 s_{1j}[V_3 = \zeta]\mathrm{v}_3^1 - \lambda^{l-p}s_{1j}^*[V_2 = 1]\mathrm{v}_2^0 - \lambda^{p-c}s_{1j}^*[V_2 = \zeta]\mathrm{v}_2^1 \quad (j = 1,3) \tag{4.6}$$

$$Rs_{12} = -\lambda^0 s_{12}^*[V_1 = 1]\mathrm{v}_1^0 - \lambda^{-l+2p-c}s_{12}^*[V_1 = \zeta]\mathrm{v}_1^1 - \lambda^{p-c}s_{12}^*[V_3 = 1]\mathrm{v}_3^0 -$$

$$-\lambda^{-l+p}s_{12}^*[V_3 = \zeta]\mathrm{v}_3^1 - \lambda^{2l-2p}s_{12}[V_2 = 1]\mathrm{v}_2^0 - \lambda^{l-c}s_{12}[V_2 = \zeta]\mathrm{v}_2^1 \tag{4.7}$$

Here the asterisks recall that the factor λ^{-l+p}, about which we have spoken above, has been taken into account in the corresponding term. Consequently, we can assume that all the quantities $s[\cdot]$ with an asterisk or without it are commensurable with λ^0.

5. THE BOUNDARY CONDITIONS ON A CLAMPED BUTT

To formulate the boundary conditions of the two-dimensional theory of shells on a clamped butt we must substitute the reactive boundary stresses, calculated from (4.6) and (4.7), into the four conditions for satisfying the modified Saint-Venant principle (3.15) with $j = 0$. The latter contain integrals which must be evaluated in the interval $(-h, +h)$ symmetrical about α_3. Hence, some of these vanish in view of the evenness or oddness of the functions $s[\cdot]$ with respect to α_3. It can be shown that these properties can be expressed as follows: s_{11}, s_{12}, s_{13} are even when, in the corresponding typical problem, the inhomogeneous butt conditions are specified by one of the three equations $V_1 = 1$, $V_2 = 1$, $V_3 = \zeta$, and odd if the last equations have the form $V_1 = \zeta$, $V_2 = \zeta$, $V_3 = 1$.

Taking this into account we obtain, from the first and fourth decay conditions (3.15), after making the replacement $\alpha_3 = h\zeta$ (the integration with respect to ζ is from –1 to 1)

$$\lambda^{l-p}\mathrm{v}_1^0\int s_{11}[V_1 = 1]d\zeta + \lambda^0\mathrm{v}_3^1\int s_{11}[V_3 = \zeta]d\zeta + \lambda^{l-p}\mathrm{v}_2^0\int s_{11}^*[V_2 = 1]d\zeta = 0 \tag{5.1}$$

$$\lambda^0\mathrm{v}_1^0\int s_{12}^*[V_1 = 1]d\zeta + \lambda^{l+p}\mathrm{v}_3^1\int s_{12}^*[V_3 = \zeta]d\zeta + \lambda^{2l-2p}\mathrm{v}_2^0\int s_{12}[V_2 = 1]d\zeta = 0$$

while the third and second conditions of (3.15) give

$$\lambda^{p-c}\mathrm{v}_1^1\int s_{11}[V_1 = \zeta]\zeta d\zeta + \lambda^{l-c}\mathrm{v}_3^0\int s_{11}[V_3 = 1]\zeta d\zeta + \lambda^{p-c}\mathrm{v}_2^1\int s_{11}^*[V_2 = \zeta]\zeta d\zeta = 0 \tag{5.2}$$

$$\lambda^{p-c}\mathrm{v}_1^1\int s_{13}[V_1 = \zeta]d\zeta + \lambda^{l-c}\mathrm{v}_3^0\int s_{13}[V_3 = 1]d\zeta + \lambda^{p-c}\mathrm{v}_2^1\int s_{13}^*[V_2 = \zeta]d\zeta = 0$$

Equations (5.1) and (5.2) form two independent systems of linear algebraic equations in (v_1^0, v_2^0) and (v_3^0, v_1^1), respectively, which, according to (1.5), are proportional to the displacements u_1, u_2 and w and the elastic angle of rotation γ_1 in the two-dimensional theory of shells. Since (5.1) and (5.2) must be satisfied on the edge $\alpha_1 = 0$ the required boundary conditions corresponding to the clamped butt are determined by these equations.

In systems (5.1) and (5.2) all the definite integrals can be regarded as known quantities. To determine them one needs to solve typical problems, in the formulation of which small parameters do not occur. Hence, we can assume that they are all commensurable with λ^0 and the asymptotic form of the coefficients of the systems discussed are explicitly expressed by the powers of λ occurring in them. Hence, one can also easily obtain the asymptotic form of the unknowns in (5.1) and (5.2). It has the form

$$v_1^0 = O(\lambda^{-l+p} v_3^1),\quad v_2^0 = O(\lambda^{-2l+2p} v_3^1),\quad v_1^1 = O(\lambda^0 v_2^1),\quad v_3^0 = O(\lambda^{-l+p} v_2^1)\quad (\text{for } \alpha_1 = 0) \tag{5.3}$$

Hence, taking (1.5) and (4.4) into account, it follows that the mathematical approach used confirms the conditions set in the two-dimensional theory from physical considerations

$$u_1 = u_2 = w = \gamma_1 = 0$$

and establishes that their error has the form $O(\lambda^{-l+p})$.

Remark. The third of estimates (5.3) does not denote "complete inaccuracy" of the boundary condition for γ_1. It must be taken into account that, by virtue of (4.4), v_2^1 vanishes when $\alpha_1 = 0$ together with the quantities v_2^0, v_3^0.

6. COMPLEXLY CLAMPED BUTTS OF THE SHELL

1. *A shell with a partially clamped butt.* Consider the butt $\alpha_1 = 0$, clamped at one part of its thickness and free along the remaining part. We will assume that the inequalities $1 \geqslant \zeta \geqslant 1 - \chi$ and the butt conditions, which follow from (1.4), correspond to the clamping, while the inequalities $1 - \chi \geqslant \zeta \geqslant -1$ and the butt conditions that follow from (1.3) correspond to the free part, where $\varkappa$ is a fixed proper fraction.

In this case we must put

$$\alpha = l,\quad \beta = l \tag{6.1}$$

Hence, we obtain the limit butt conditions on the fixed part of the end

$$\lambda^{l-p}(v_1^0 + \lambda^{-l+2p-c}\zeta v_1^1) + RV_1(\beta) = 0,\quad \lambda^{l-p}(v_2^0 + \lambda^{-l+2p-c}\zeta v_2^1) + RV_2(\alpha) = 0 \tag{6.2}$$

$$\lambda^{l-c}(v_3^0 + \lambda^{-l+c}\zeta v_3^1) + RV_3(\beta) = 0$$

and on the free part of the butt

$$\tau_{11}^0 + \lambda^{-l+2p-c}\zeta\tau_{11}^1 + ES_{11}(\beta) = 0,\quad \tau_{12}^0 + \lambda^{-l+2p-c}\zeta\tau_{12}^1 + ES_{12}(\alpha) = 0 \tag{6.3}$$

$$\lambda^{-l+p}(\tau_{13}^0 + \zeta\tau_{13}^1 + \lambda^{-l+2p-c}\zeta^2\tau_{13}^2) + ES_{13}(\beta) = 0$$

(We have taken into account the fact that, by virtue of (1.9) and (1.10), the inequality $-2l + 3p - c < 0$ holds.)

The necessary requirements, which ensure the satisfaction of the limit butt conditions, formulated in Section 2, are satisfied, if relations of the form (5.3), obtained for butt clamped over the whole thickness, are realizable with sufficient asymptotic accuracy. Hence, also, from considerations presented below, it follows that in the classical two-dimensional theory of shells the same boundary conditions are established on a partially clamped butt as on butt which is clamped as a whole. On the free part of the butt discrepancies arise which can obviously be removed when solving the problem of the boundary stress–strain state for the half-strip shown in Fig. 1.

The solution of this problem (without having to satisfy any additional conditions) will always have a decaying form, which arise from the following physical considerations.

In the clamped part of the butt of the half-strip reactions can, in principle, occur, which balance any forces applied to its free part. Hence, the assumption that the solution decays in this case is not in clear

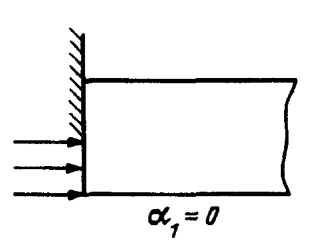

Fig. 1.

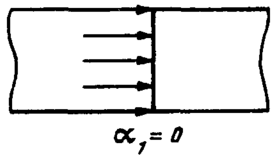

Fig. 2.

contradiction with the requirements of global balancing of the strip. At the same time the Saint-Venant principle (including the modified principle) can be treated as an assertion that the stress–strain state will always decay whenever this does not give rise to global unbalancing.

The unconditional decay of the boundary stress–strain state in this case can also be revealed by the formal discussions used in Section 5 for a completely clamped butt. To do this we must introduce the idea of typical problems, which enable the reactive forces which arise from the butt conditions (6.2) to be expressed in terms of their solutions, and to relate to these actions forces which directly arise from conditions (6.3). Then, the four requirements of the applicability of the modified Saint-Venant principle lead, as in Section 5, to four linear algebraic equations, defining the boundary values of υ and τ. Powers of λ, contained in (6.2) and (6.3), occur in these relations, and an asymptotic analysis of this system becomes possible. It shows that the τ-terms are asymptotically negligible compared with the υ-terms, and this is equivalent to the conclusion drawn above (it is also indirectly confirmed by the asymptotic structure of the butt relations mentioned).

We will not dwell on the details of this kind of discussion here or below. We will merely note two facts: (1) in the classical theory they in no way lead to a disparity between the number of boundary conditions and the possibility of satisfying them; (2) it is always implicitly assumed in them that the number $\varkappa$, which specifies the relative thickness of the clamped part, does not approach asymptotically either to zero or to unity (otherwise, to formulate the boundary conditions, a preliminary solution of typical problems of the theory of the boundary stress–strain state would be required).

2. *The junction between two shells of different thickness.* In this case, for $\alpha_1 = 0$ and any ζ we must satisfy all six equations, which follow from (1.3) and (1.4), but we must replace the quantities τ, υ, S and V in them by the jumps which they undergo on passing through $\alpha_1 = 0$.

In these problems formulae (6.1) for α, β remain true. Taking this into account, and setting up the six limit conditions at the junction, it can be shown that in the corresponding kinematic relations, i.e. in the analogue of (6.2), we must put

$$\delta v_1^0 = \delta v_2^0 = \delta v_3^0 = \delta v_1^1 = 0 \tag{6.4}$$

in order that all the positive powers of λ in them disappear and the term $\lambda^0 \zeta \delta v_3^1$, which ensures inhomogeneity of the problem of determining the boundary stress–strain state, remains.

This problem must be solved in the strip shown in Fig. 2. The external forces, applied to the joint $\alpha_1 = 0$ of the strip, are determined by terms with the quantities $\delta\tau$ in analogues of Eqs (6.3). These forces must be subject to the conditions of Saint-Venant decay. In this case (when $\alpha = \beta$) they have the following form in the notation of formulae (1.5)

$$\delta T_1 = \delta S_{21} = \delta N_1 = \delta G_1 = 0 \quad (\text{when } \alpha_1 = 0) \tag{6.5}$$

It follows from (6.4) and (6.5) that the treatment of the conditions at the junction of two shells that is traditional in the two-dimensional classical theory, is largely confirmed. Moreover, conditions (6.5) show that the Kelvin–Tait transformation in this case leads to an error: at the junction it is the jump in the "real" rather than the "reduced" shearing force that should vanish.

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